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LETTER TO THE EDITOR

# Lie symmetries and Painlevé test for explicitly space- and time-dependent nonlinear wave equations

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**Abstract.** We investigate the Lie symmetry vector fields of the wave equation $\Box u+f(x_1,\ldots,x_n,u)=0$ where $f$ is some nonlinear smooth function and $n\geqslant 2$. The Painlevé test is considered for the construction of explicitly space- and time-dependent integrable one-space-dimensional nonlinear wave equations.

The construction of Lie symmetry vector fields and the associated Lie transformation group that leaves a partial differential equation invariant is well documented in the literature (Ovsiannikov 1982, Olver 1986, Fushchich *et al* 1989, Bluman and Kumei 1989, Euler and Steeb 1992).

In this letter we investigate the equation

$$\Box u+f(x_1,\ldots,x_n,u)=0 \tag{1}$$

where

$$\Box:=\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}-\frac{\partial^2}{\partial x_n^2} \tag{2}$$

with $x_n\equiv t$ and $n\geqslant 2$. The underlying metric tensor field is given by

$$g=\sum_{i=1}^{n-1}\mathrm{d}x_i\otimes\mathrm{d}x_i-\mathrm{d}x_n\otimes\mathrm{d}x_n\,. \tag{3}$$

Equation (1), with $f=0$, can be derived from

$$\mathrm{d}(*\mathrm{d}u)=0$$

where d is the exterior derivative and $*$ the Hodge-star operator (Euler and Steeb 1992).

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The conformal vector fields $\mathcal{Z}$ for the metric tensor field (3) on $\mathcal{R}^n$ form an $(n+1)(n+2)/2$-dimensional Lie algebra. These vector fields are defined by

$$L_Z g = \rho_Z g$$

for some $C^\infty$ function $\rho_Z$ where $L_Z(\cdot)$ denotes the Lie derivative. The basis of the Lie algebra is spanned by

$$\mathcal{T} = \frac{\partial}{\partial x_4} \qquad \rho_\mathcal{T} = 0$$

$$\mathcal{P}_i = \frac{\partial}{\partial x_i} \qquad \rho_{\mathcal{P}_i} = 0 \qquad i = 1, 2, \ldots, n-1$$

$$\mathcal{R}_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \qquad \rho_{\mathcal{R}_{ij}} = 0 \qquad i \neq j, \quad i, j = 1, 2, \ldots, n-1$$

$$\mathcal{L}_i = x_i \frac{\partial}{\partial x_n} + x_n \frac{\partial}{\partial x_i} \qquad \rho_{\mathcal{L}_i} = 0 \qquad i = 1, 2, \ldots, n-1$$

$$\mathcal{S} = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} \qquad \rho_\mathcal{S} = 2$$

$$\mathcal{I}_i = 2x_i \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} - \left( \sum_{k=1}^{n-1} x_k^2 - x_n^2 \right) \frac{\partial}{\partial x_i} \qquad \rho_{\mathcal{I}_i} = -4x_i \qquad i = 1, 2, \ldots, n-1$$

$$\mathcal{I}_n = 2x_n \sum_{i=1}^{n-1} x_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} x_j^2 \frac{\partial}{\partial x_n} \qquad \rho_{\mathcal{I}_n} = 4x_n\,.$$

Note that the vector fields $\mathcal{T}$, $\mathcal{P}_i$, $\mathcal{R}_{ij}$ and $\mathcal{L}_i$ are the Killing vector fields for (3). The physical interpretation of the given vector fields is the following: $\mathcal{T}$ generates time translation; $\mathcal{P}_i$ space translation; $\mathcal{R}_{ij}$ space rotation; $\mathcal{L}_i$ space–time rotations, and $\mathcal{S}$ the uniform dilatations or scaling $(x_1, \ldots, x_n) \mapsto \varepsilon(x_1, \ldots, x_n)$ where $\varepsilon > 0$. $\mathcal{I}_n$ is the conjugation of $\mathcal{T}$ by the inversion in the unit hyperboloid $Q : (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n)/(\sum_{i=1}^{n-1} x_i^2 - x_n^2)$, and the $\mathcal{I}_i$ are the conjugations of the $\mathcal{P}_i$ by $Q$.

For the case $n = 2$ we now consider the explicitly space- and time-dependent equation

$$\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} + f(x_1, x_2, u) = 0 \tag{4}$$

where $x_2 \equiv t$. We can state the following.

*Theorem.* Equation (4) admits the Lie symmetry vector field

$$Z = \xi_1(x_1, x_2)\frac{\partial}{\partial x_1} + \xi_2(x_1, x_2)\frac{\partial}{\partial x_2} + \eta(x_1, x_2)\frac{\partial}{\partial u}$$

with

$$\xi_1(x_1, x_2) = -b_1(x_1^2 - x_2^2) + 2x_1(b_1x_1 - b_2x_2) + cx_2$$
$$\xi_2(x_1, x_2) = -b_2(x_1^2 - x_2^2) + 2x_2(b_1x_1 - b_2x_2) + cx_1$$
$$\eta(x_1, x_2) = (\lambda_1(x_1^2 - x_2^2)^{-1} + \lambda_2)(b_1x_1 - b_2x_2)$$

if and only if

$$f(x_1, x_2, u) = (x_1^2 - x_2^2)^{-2} g\left(\frac{\lambda_1}{2}(x_1^2 - x_2^2)^{-1} - \frac{\lambda_2}{2}\ln(x_1^2 - x_2^2) + u\right) \tag{5}$$

where $g$ is an arbitrary smooth function of its argument and $b_1, b_2, c, \lambda_1, \lambda_2 \in \mathcal{R}$.

*Proof.* In the 2-jet bundle $J^2(\mathcal{R}^3)$, equation (4) determines a submanifold given by the constrained equations

$$F \equiv u_{11} - u_{22} + f(x_1, x_2, u) = 0\,. \tag{6}$$

The invariance condition is given by $L_{\bar{Z}} F \hat{=} 0$ where $\bar{Z}$ is the prolonged vector field and $\hat{=}$ stands for the restriction to the solution of (6) and its prolongations. For the given Lie symmetry vector field we obtain the condition on $f$ as

$$2\omega\frac{\partial f}{\partial \omega} + (\lambda_1\omega^{-1} + \lambda_2)\frac{\partial f}{\partial \omega} + 4f = 0$$

where $\omega = x_1^2 - x_2^2$, with the general solution given by (5).

Equation (4) thus admits, in addition to the rotation $\mathcal{L}_1$, the field-extended conjugations

$$I_1 = (x_1^2 + x_2^2)\frac{\partial}{\partial x_1} + 2x_1x_2\frac{\partial}{\partial x_2} + x_1(\lambda_1(x_1^2 - x_2^2)^{-1} + \lambda_2)\frac{\partial}{\partial u}$$
$$I_2 = 2x_1x_2\frac{\partial}{\partial x_1} + (x_1^2 + x_2^2)\frac{\partial}{\partial x_2} + x_2(\lambda_1(x_1^2 - x_2^2)^{-1} + \lambda_2)\frac{\partial}{\partial u}\,.$$

Note that the vector fields $\langle \mathcal{L}_1, I_1, I_2\rangle$ form a three-dimensional Lie algebra with commutation relation

$$[\mathcal{L}_1, I_1] = I_2 \qquad [\mathcal{L}_1, I_2] = I_1 \qquad [I_1, I_2] = 0$$

if and only if $\lambda_1 = 0$. This condition is imposed by the commutation relations.

Before we apply the Painlevé test we briefly describe the approach used. The conjecture is that a nonlinear partial differential equation is integrable (by the inverse-scattering transform, Bäcklund transformation, or some other method), if and only if it possesses the Painlevé property (possibly after a change of variables)—see Steeb and Euler (1988), Ward (1988) and Weiss (1986) for more details. The formulation is as follows: Suppose we have a nonlinear partial differential equation with analytic coefficients (or a set of $M$ partial differential equations for $M$ functions, denoted collectively by $u$). Complexify the functions, so that we now have $M$ holomorphic functions with complex independent variables. Let $\phi$ be a holomorphic function such that $\phi = 0$ is not a characteristic hypersurface for the partial differential equation. The power series

$$u = \phi^{-\alpha} \sum_{j=0}^{\infty} u_j \phi^j \tag{7}$$

is then considered where $\alpha$ is a suitable positive integer which is given by the dominant behaviour of the equation. This power series must then be substituted into the partial differential equation. This leads to recursion relations between the $u_j$. The Painlevé property states that the recursion relations should be consistent, and that the expansion (7) should contain the maximum number of arbitrary functions (counting $\phi$ as one of them). We also say that the partial differential equation passes the Painlevé test. The position in the expansion where the arbitrary functions should occur are called the resonances.

We now apply the Painlevé test to the wave equation (4) with the function (5) where $\lambda_1 = \lambda_2 = 0$. For the purpose of this study we consider the explicitly space- and time-dependent sine-Gordon equation, Liouville equation, and Mikhailov equation which are special cases of (4) with the function (5). We introduce arbitrary space- and time-dependent functions in the above equations. Application of the Painlevé test then results in conditions on these arbitrary functions. The relation to (4) (together with the function (5)) and its symmetries is discussed.

*Case 1.* We consider the explicitly space- and time-dependent sine-Gordon equation

$$\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} + h(x_1, x_2) \sin u = 0 \tag{8}$$

where $h$ is a smooth function. It is well known that the above one-dimensional sine-Gordon equation, with $h = 1$, is integrable by the inverse scattering transform and that it passes the Painlevé test after the transformation of the dependent variable, i.e. $v = \exp(iu)$. Applying this transformation to (8) gives

$$v\frac{\partial^2 v}{\partial x_1^2} - \left(\frac{\partial^2 v}{\partial x_1^2}\right)^2 - v\frac{\partial^2 v}{\partial x_2^2} + \left(\frac{\partial v}{\partial x_2}\right)^2 + \frac{1}{2}h(x_1, x_2)(v^3 - v) = 0\,. \tag{9}$$

We now ask the following question: is there a non-trivial smooth function $h$ such that (9) passes the Painlevé test? Inserting the ansatz $v \propto v_0\phi^m$ we find $\alpha = -2$ and

$$v_0 = 4\left(\left(\frac{\partial\phi}{\partial x_2}\right)^2 - \left(\frac{\partial\phi}{\partial x_1}\right)^2\right)h^{-1}.$$

The resonances are $\{-1, 2\}$. Inserting expansion (7) into (9) yields

$$v_1 = 4\left(\frac{\partial^2\phi}{\partial x_1^2} - \frac{\partial^2\phi}{\partial x_2^2}\right)h^{-1}.$$

At the resonance 2 we obtain

$$\left(\frac{\partial^2 h}{\partial x_2^2}h - \left(\frac{\partial h}{\partial x_2}\right)^2 - \frac{\partial^2 h}{\partial x_1^2}h + \left(\frac{\partial h}{\partial x_1}\right)^2\right)\left(\left(\frac{\partial\phi}{\partial x_1}\right)^2 - \left(\frac{\partial\phi}{\partial x_2}\right)^2\right)^2 = 0\,.$$

We thus obtain the constraint on $h$, namely

$$\left(\frac{\partial h}{\partial x_2}\right)^2 - \frac{\partial^2 h}{\partial x_2^2} h = \left(\frac{\partial h}{\partial x_1}\right)^2 - \frac{\partial^2 h}{\partial x_1^2} h \tag{10}$$

with the general solution

$$h(x_1, x_2) = h_1(x_1 + x_2) h_2(x_1 - x_2) . \tag{11}$$

Here $h_1$ and $h_2$ are arbitrary smooth functions. Solution (11) follows from the fact that (10) can be transformed to the linear wave equation

$$\frac{\partial^2 \tilde{h}}{\partial x_1^2} = \frac{\partial^2 \tilde{h}}{\partial x_2^2}$$

by applying $h = \exp(\tilde{h})$. We thus find that

$$\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} + h_1(x_1 + x_2) h_2(x_1 - x_2) \sin u = 0 \tag{12}$$

passes the Painlevé test. Equation (12) is integrable since it can be transformed to the Sine-Gordon equation

$$\frac{\partial^2 \bar{u}}{\partial \xi \partial \eta} = \frac{1}{4} \sin \bar{u}$$

by the transformation (Steeb 1992)

$$\xi(x_1, x_2) = \int^{x_1 + x_2} f_1(s)\, \mathrm{d}s \qquad \eta(x_1, x_2) = \int^{x_1 - x_2} f_2(s)\, \mathrm{d}s$$
$$\bar{u}(\xi(x_1, x_2), \eta(x_1, x_2)) = u(x_1, x_2) .$$

Equation (12) is a special case of (4), with function (5), where

$$g(u) = \sin u \tag{13}$$

$$h_1 = \frac{1}{(x_1 + x_2)^2} \qquad h_2 = \frac{1}{(x_1 - x_2)^2} . \tag{14}$$

Equation (12) thus admits the symmetry vector fields $\mathcal{L}_1$, $I_1$ and $I_2$ (with $\lambda_1 = \lambda_2 = 0$).

*Case 2.* We consider the explicitly space- and time-dependent Liouville equation

$$\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} + h(x_1, x_2) \mathrm{e}^u = 0 . \tag{15}$$

In the case where $h = 1$, (15) is integrable and passes the Painlevé test. Performing the Painlevé test for the above equation we find that (15) passes the Painlevé test if $h$ satisfies the condition given by (10). This relates to (4) and function (5) (together with its symmetries) with $g(u) = \exp(u)$ and $h_1$, $h_2$ given by (14).

*Case 3.* Finally we consider the explicitly space- and time-dependent Mikhailov equation

$$\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} = g_1(x_1, x_2)\,\mathrm{e}^u + g_2(x_1, x_2)\,\mathrm{e}^{-2u} \,. \tag{16}$$

For $h_1 = h_2 = 1$ the Mikhailov equation is integrable by a $3 \times 3$ matrix scattering problem. With $v = \exp u$ (16) takes the form

$$u\left(\frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2}\right) + \left(\frac{\partial u}{\partial x_2}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 = g_1(x_1, x_2)u^3 + g_2(x_1, x_2) \,. \tag{17}$$

Equation (17) has two branches. For the first branch the dominant behaviour is given by $\alpha = -2$ and the resonances are $\{-1, 2\}$. The equation passes the Painlevé test (for this branch) if $g_1$ satisfies condition (10). No condition is imposed on function $g_2$. For the second branch we find $\alpha = 1$ and the resonances are $\{-1, 2\}$. This branch imposes a condition on function $g_2$, namely (10). No condition is imposed on $g_1$. Since both branches have to be taken into account, we conclude that $g_1$ and $g_2$ have to satisfy condition (10). This relates to (4) and function (5) (together with its symmetries) with $g(u) = \exp(u) + \exp(-2u)$ and $h_1$, $h_2$ given by (14).

In the above three cases we note that if the equations admit the Lie symmetry vector fields $\mathcal{L}_1$, $I_1$ and $I_2$ (with $\lambda_1 = \lambda_2 = 0$), the equations also pass the Painlevé test.

Let us now consider the case $n \geqslant 3$. The case $n = 4$ was studied by Euler and Steeb (1989). It can be shown (see for example Ibragimov 1972) that for $n \geqslant 3$ the equation

$$\Box u - \lambda u^{(n+2)/(n-2)} = 0 \tag{18}$$

($\lambda \in \mathcal{R}$) admits, in addition to translations $\mathcal{T}$, $\mathcal{P}_i$ and rotations $\mathcal{R}_{ij}$, $\mathcal{L}_i$, the Lie symmetry vector fields

$$S = \mathcal{S} - \left(\frac{n-2}{2}\right) u\frac{\partial}{\partial u} \qquad I_i = \mathcal{I}_i - (n-2)x_i u\frac{\partial}{\partial u} \qquad I_n = \mathcal{I}_n - (n-2)x_n u\frac{\partial}{\partial u}$$

where $\mathcal{S}$, $\mathcal{I}_i$ and $\mathcal{I}_n$ are the conformal vector fields of the metric tensor field (3) and $i \neq j = 1, \ldots, n-1$.

Equation (18) has the Hamiltonian density

$$\mathcal{H}(u, p_i) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \left(\frac{n-2}{2n}\right) u^{2n/(n-2)} \,.$$

Owing to the scaling vector field $S$ we know that (18) is scale-invariant under

$$x_i \mapsto \varepsilon^{-1} x_i \qquad u \mapsto \varepsilon^{(n-2)/2} u \,.$$

Therefore the Hamiltonian density scales as

$$\mathcal{H}(\varepsilon^{(n-2)/2} u \qquad \varepsilon^{n/2} p_i) = \varepsilon^n \mathcal{H}(u, p_i)$$

so that two resonances (also known as Kowalevski exponents) are given by $-1$ and $n$. The resonance $-1$ is associated with the function $\phi$.

For $n = 3$ and $n = 4$ the condition at the resonances are not satisfied. We consider the following transformation of the dependent variable:

$$u(x_1, \ldots, x_n) = v^{n-2}(x_1, \ldots, x_n)$$

for the case $n \geqslant 4$. Equation (18) then takes the form

$$\sum_{j=1}^{n-1}\left((n-2)(n-3)v^{n-4}\left(\frac{\partial v}{\partial x_j}\right)^2 + (n-2)v^{n-3}\frac{\partial^2 v}{\partial x_j^2}\right) - (n-2)(n-3)v^{n-4}\left(\frac{\partial v}{\partial x_n}\right)^2$$
$$- (n-2)v^{n-3}\frac{\partial^2 v}{\partial x_n^2} + \lambda v^{n+2} = 0\,. \tag{19}$$

For the dominant behaviour of (19) we find that $\alpha = \frac{1}{2}$. This indicates that (18) also does not pass the Painlevé test for $n \geqslant 4$.

Let us mention that within the jet-bundle approach exact solutions for (1) can be constructed by considering (Euler and Steeb 1992)

$$js^*(Z \lrcorner \theta) = 0$$

where $\theta := \mathrm{d}u - \sum_{j=1}^{n} u_i \mathrm{d}x_i$ is the contact form on the jet bundle $J^1(E)$ and $Z$ is a Lie symmetry vector field (or a linear combination of Lie symmetry vector fields) admitted by (1).

## References


Bluman G W and Kumei S 1989 *Symmetries and Differential Equations (Applied Mathematical Sciences 81)* (New York: Springer)

Euler N and Steeb W-H 1989 *Int. J. Theor. Phys.* **28** 11

—— 1992 *Continuous Symmetries, Lie Algebras and Differential Equations* (Mannheim: Wissenschaftsverlag)

Fushchich W I, Shtelen W M and Serov N I 1989 *Symmetry Analysis and Exact Solutions for Nonlinear Equations in Mathematical Physics* Academy of Science of the Ukraine, Kiev (in Russian) (English version 1992 (Deventer: Kluwer))

Ibragimov N H 1972 *Lie Groups in some Problems of Mathematical Physics* Novosibirk University (in Russian)

Olver P J 1986 *Applications of Lie Groups to Differential Equations* (New York: Springer)

Ovsiannikov L V 1982 *Group Analysis of Differential Equations* ed W F Ames (New York: Academic) (Translation from Russian)

Steeb W-H 1992 *Problems in Theoretical Physics vol II: Advanced Problems* (Mannheim: Wissenschaftsverlag)

Steeb W-H and Euler N 1988 *Nonlinear Evolution Equations and Painlevé Test* (Singapore: World Scientific)

Ward R S 1988 *Nonlinearity* **1** 671

Weiss J 1986 *J. Math. Phys.* **27** 1293